Exam 2, Biostatistics 341

31 October, 2011

Please show all your work and perform all calculations to whatever degree of exactness you are able. This test is closed book and no calculators are allowed.

1. (20) Let X, Y, and Z be random variables and a, b, and c constants.

(10) a. Starting with the definition of covariance, prove that $Cov(aX + bY, cZ) = acCov(X, Z) + bcCov(Y, Z)$.

(5) b. What is $Cov(aX + bY, cZ)$ if Y and Z are independent?

(5) c. What is the correlation between aX and cZ?

2. (20) Let X follow a Poisson(λ) distribution. Let $Y|X$ be normally distributed with mean $\beta_0 + \beta_1 X$ and variance σ^2.

(8) a. What is the joint distribution (pdf) of X, Y? (no need to simplify)

(6) b. What is $E(Y)$?

(6) c. What is $Var(Y)$?

3. (60) Let X and Y be independent random variables that follow an exponential distribution with mean 1. Let $U = X/(X + Y)$ and $V = X + Y$.

(7) a. What is $P(X > Y)$?

(9) b. What is the joint distribution (pdf) of U, V?

(7) c. What is the marginal distribution (pdf) of U? Give its name.

(7) d. What is the marginal distribution (pdf) of V? Give its name.

(5) e. What is the conditional distribution of U given V?

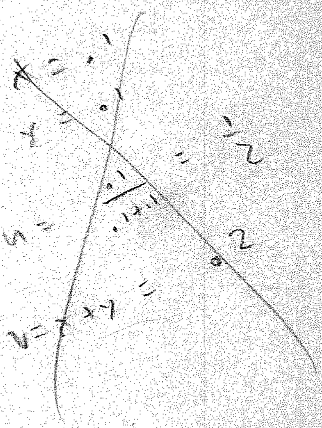
(5) f. What is the conditional distribution of V given U?

(5) g. Is V a member of an exponential family distribution? Why or why not?

(5) h. Are U and V independent? Why or why not?

(5) i. What is the distribution of $Z = \sigma U + \mu$, where $\sigma > 0$ and μ are constants?

(5) j. What is greater, $E(V^{1/2})$ or $(E(V))^{1/2}$? Why?



$\boxed{1}$ $\boxed{a}$ $Cov(aX+bY, cZ) = E((aX+bY)cZ) - E(aX+bY)E(cZ)$

$= E(acXZ + bcZY) - (E(aX) + E(bY))E(cZ)$

$= acE(XZ) + bcE(ZY) - acE(X)E(Z) - bcE(Y)E(Z)$

$= ac\,Cov(X,Z) + bc\,Cov(Y,Z)$ ∎

$\boxed{B}$ $Cov(aX+bY, cZ) =$

$ac\,Cov(X,Z) + 0 = ac\,Cov(X,Z).$

$\boxed{c}$ $$corr(aX, cZ) = \frac{Cov(aX, cZ)}{\sqrt{Var(aX)\,var(cZ)}}$$

$$= \frac{ac\,Cov(X,Z)}{|ac|\sqrt{Var(X)\,Var(Z)}}$$

$$= \begin{cases} \dfrac{Cov(X,Z)}{\sqrt{Var(X)\,Var(Z)}} & \text{if } ac > 0 \\ \dfrac{-Cov(X,Z)}{\sqrt{Var(X)\,Var(Z)}} & \text{if } ac < 0 \\ \text{undefined} & \text{if } ac = 0 \end{cases}$$

~~(3:00)~~

2

a

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$f(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - \beta_0 - \beta_1 x)^2\right)$$

$$\Rightarrow \quad f(x,y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\beta_0-\beta_1 x)^2\right) \frac{e^{-\lambda}\lambda^x}{x!}$$

for $x = 0, 1, \ldots\ldots$
$-\infty < y < \infty$

b

$$E(Y) = E(E(Y|X)) = E(\beta_0 + \beta_1 X) = \beta_0 + \beta_1 E(X) = \boxed{\beta_0 + \beta_1 \lambda}$$

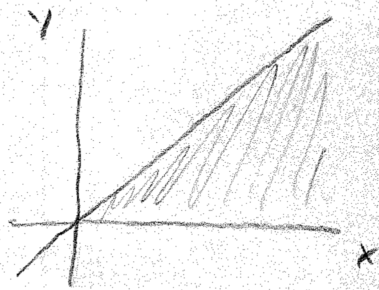
c

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = E(\sigma^2) + Var(\beta_0 + \beta_1 X) = \sigma^2 + \beta_1^2 Var(X) = \boxed{\sigma^2 + \beta_1^2 \lambda}$$

~~(13:00)~~

$\boxed{3}$ $f(x) = e^{-x} \quad x > 0$

$f(y) = e^{-y} \quad y > 0$



$\boxed{a}$ $P(X > Y) = \displaystyle\int_0^{\infty}\int_0^{x} e^{-x} e^{-y} \, dy \, dx$

$$= \int_0^{\infty} e^{-x} \, e^{-y}(-1)\Big]_0^{x} dx$$

$$= \int_0^{\infty} (e^{-x})(1 - e^{-x}) \, dx$$

$$= -e^{-x} + \frac{1}{2} e^{-2x}\Big]_0^{\infty} = 0 + 1 + \frac{1}{2}(0 - 1)$$

$$= 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

$\boxed{b}$ $u = \dfrac{x}{x+y} \qquad v = x + y$

$\Rightarrow u = \dfrac{x}{v} \Rightarrow x = uv \qquad \Rightarrow \dfrac{\partial x}{\partial u} = v \qquad \dfrac{\partial x}{\partial v} = u$

$y = v - uv \qquad \dfrac{\partial y}{\partial u} = -v \qquad \dfrac{\partial y}{\partial v} = 1 - u$

$\Rightarrow J = v(1-u) + uv$

$= v - uv + uv$

$= v$

$\Rightarrow f_{u,v}(u,v) = f_{x,y}(x,y)\,|J|$

$= e^{-uv} e^{-(v-uv)} \, v$

$= e^{-uv - v + uv} \, v$

$= ve^{-v}$ for $v > 0, \; 0 < u < 1$

Points
2 for plugging in
1 for range
2 for exponential
2 right answer
1 for Jacobian

gamma distribution

~~$f(u) = \int f(u,v)\,dv$~~

~~3 =~~

~~right answer = 2~~

~~Name = 2~~

$\boxed{c}$ $f_u(u) = \int_0^\infty v e^{-v}\,dv = 1$ for $0 < u < 1$

$\Rightarrow$ $U \sim \text{Uniform}(0,1)$

$\boxed{d}$ $f_v(v) = \int_0^1 v e^{-v}\,du = u v e^{-v} \Big]_0^1 = v e^{-v}$ for $v > 0$

$\Rightarrow$ $V \sim \text{gamma}(2,1)$

$\boxed{e}$ $f(u|v) = \dfrac{f(u,v)}{f(v)} = \dfrac{ve^{-v}}{ve^{-v}} = 1$ for $0 < u < 1$

$\boxed{f}$ $f(v|u) = \dfrac{ve^{-v}}{1} = ve^{-v}$ for $v > 0$

~~2 correct~~

~~3 justification~~

$\boxed{g}$ Yes. $f(v) = \underbrace{v}_{h(v)}\exp\{\underbrace{-}_{w(\theta)}\underbrace{v}_{t(v)}\}\underbrace{\cdot 1}_{c(\theta)}$

$\boxed{h}$ Yes, $U \perp V$ because $f(v) = f(v|u)$.

Also $f(u,v) = f_u(u) f_v(v)$.

$\boxed{i}$ $Z = \sigma U + \mu$

$\Rightarrow$ $Z \sim \text{Unif}(\mu, \mu + \sigma)$

$\boxed{j}$ $g(v) = v^{1/2} \Rightarrow g'(v) = \frac{1}{2}v^{-1/2} \Rightarrow g''(v) = -\frac{1}{4}v^{-3/2} < 0 \Rightarrow$ concave

$\Rightarrow E(\sqrt{V}) < \sqrt{E(V)}$