Exam 1, Biostatistics 341
26 September, 2011

Please show all your work and perform all calculations to whatever degree of exactness you are able. This test is closed book and no calculators are allowed.

1. (20) Five fair dice (6-sided with equal probability for each side) are rolled.
   a. What is the probability that none of the dice shows a 6?
   b. What is the probability that all 5 dice show a different number?
   c. What is the probability of exactly two 6s?
   d. If the dice are rolled sequentially until a 6 is rolled, what is the probability that this will occur on the 5th role?

2. (30) Let X have the density $f_X(x) = \alpha/(1+x)^{1+\alpha}$ for $x \geq 0$, for some value of $\alpha > 0$.
   a. What is the density of Y =log$(1+X)$.
   b. Evaluate $F_X(x)$ for $x \geq 0$.
   c. Evaluate $F_X^{-1}(t)$ for $0 < t < 1$.
   d. Suppose X and Z are independent, and both have the density $f_X(x)$ given above. What is $P(\min(X, Z) \leq x)$?

3. (30) Let $X \sim \text{GAM}(\alpha, \beta)$; in other words, $f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ for $x > 0, \alpha > 0, \beta > 0$.
   a. What is the moment generating function for X?
   b. What is the mean of X? (evaluate either using the mgf or directly)
   c. What is the mean of $5X$+3?
   d. What is the variance of X?
   e. What is the variance of $5X$+3?

4. (20) In a randomized trial, the probability of infection if randomized to control is 0.6. The probability of infection if randomized to treatment is 0.5. Two-thirds of patients are randomized to treatment, one-third are randomized to control.
   a. Given someone is infected, what is the probability that they were randomized to treatment?
   b. What are bounds on the probability of being infected no matter what treatment a participant is assigned? (i.e., Define S_0 as the indicator of infection if assigned to control (i.e., $S_0 = 1$ if infected under control and $S_0 = 0$ if not infected under control) and S_1 as the indicator of infection if assigned to placebo. Only one of S_0 and S_1 is observed. We know $P(S_0 = 1) = 0.6$ and $P(S_1 = 1) = 0.5$. What are bounds for $P(S_0 = 1, S_1 = 1)$?)
   c. Assume S_0 and S_1 are independent. Then what is $P(S_0 = 1, S_1 = 1)$? Do you think it is realistic for S_0 and S_1 to be independent? Why or why not?

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Exam 1

1 $p = \frac{1}{6}$, $n = 5$, success = 6 rolled, X = # of 6s

(5) a $P(X=0) = \left(1-\frac{1}{6}\right)^5 = \boxed{\left(\frac{5}{6}\right)^5}$

(5) b P(all 5 dice show different #)

$= 1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} = \boxed{\frac{6!}{6^5}} = \frac{5!}{6^4}$

P(1st shows unique #) P(2nd shows unique # conditional on first) P(3rd shows unique # conditional on first 2) ... etc

(5) c $P(X=2) = \binom{5}{2} p^2 (1-p)^{5-2} = \boxed{\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3}$

(5) d $X \sim GEO(p)$

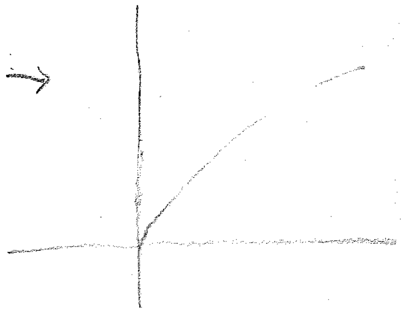
$P(X=5) = p(1-p)^4 = \boxed{\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4}$

one success

4 failures

$\boxed{2}$ $f(x) = \alpha(1+x)^{-1-\alpha}$ for $x \geq 0$ $\alpha > 0$

$\boxed{a}$ $y = \log(1+x)$ $\leftarrow$ 1 to 1 function looks something like this $\Rightarrow$



$\Rightarrow e^y = 1 + x$

$\Rightarrow x = e^y - 1 = g^{-1}(y)$

$\Rightarrow \frac{\partial g^{-1}(y)}{\partial y} = e^y$

$\Rightarrow f(y) = \alpha(1 + e^y - 1)^{-1-\alpha} e^y$

$= \alpha(e^y)^{-1-\alpha+1}$

$= \boxed{\alpha e^{-\alpha y} \quad \text{for } y \geq 0}$.

$\boxed{b}$ $F(x) = \int_0^x \alpha(1+t)^{-1-\alpha}\, dt$

$= \left. \frac{\alpha(1+t)^{-\alpha}}{-\alpha} \right|_0^x$

$= \left. -(1+t)^{-\alpha} \right|_0^x$

$= -(1+x)^{-\alpha} + (1+0)^{-\alpha}$

$= \boxed{1 - (1+x)^{-\alpha} \quad \text{for } x \geq 0}$

$\boxed{c}$ $F_X^{-1}(t) = ?$

$t = 1 - (1+x)^{-\alpha}$

$\Rightarrow t - 1 = -(1+x)^{-\alpha}$

$\Rightarrow 1 - t = (1+x)^{-\alpha}$

$\cancel{\Rightarrow (1-t)^{\alpha} = 1+x}$

$\cancel{\Rightarrow x = \boxed{(1-t)^{\alpha} - 1 = F_X^{-1}(t)}}$

$\Rightarrow (1-t) = \left(\frac{1}{1+x}\right)^{\alpha}$

$\Rightarrow (1-t)^{1/\alpha} = \frac{1}{1+x}$

$\Rightarrow 1+x = (1-t)^{-1/\alpha}$

$\Rightarrow x = (1-t)^{-1/\alpha} - 1$

$\Rightarrow F_X^{-1}(t) = \left(\frac{1}{1-t}\right)^{1/\alpha} - 1$

for $0 < t < 1$

-2

$\boxed{d}$ $X \perp Z$

$$
\begin{aligned}
P(\min(X,Z) \le x) &= 1 - P(\min(X,Z) > x) \\
&= 1 - P(X > x, Z > x) \\
&= 1 - P(X > x)\,P(Z > x) \\
&= 1 - (1 - F_X(x))(1 - F_X(x)) \\
&= 1 - (\cancel{1} - \cancel{1} + (1+x)^{-\alpha})(\cancel{1} - \cancel{1} + (1+x)^{-\alpha}) \\
&= \boxed{1 - (1+x)^{-2\alpha}}
\end{aligned}
$$

$\boxed{3}$ $f(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

$\boxed{a}$
$$M_x(t) = E(e^{tx})$$
$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$$
$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)}\, dx$$
$$= \frac{1}{\beta^\alpha}\left(\frac{\beta}{1-t\beta}\right)^\alpha \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha)\left(\frac{\beta}{1-t\beta}\right)^\alpha} x^{\alpha-1} e^{-x\left(\frac{1-t\beta}{\beta}\right)}}_{1}$$
$$= \boxed{(1-t\beta)^{-\alpha}}$$

$\boxed{b}$
$$E(x) = M_x'(t)\Big|_{t=0}$$
$$= +\alpha(1-t\beta)^{-\alpha-1}\beta\Big|_{t=0}$$
$$= +\alpha(1-0)\beta$$
$$= \boxed{\alpha\beta}$$

$\boxed{c}$ $E(5x+3) = 5E(x) + 3 = \boxed{5\alpha\beta + 3}$

$\boxed{d}$
$$E(x^2) = M_x''(t)\Big|_{t=0}$$
$$= \alpha\beta(-\alpha-1)(1-t\beta)^{-\alpha-2}(-\beta)\Big|_{t=0}$$

$$= \alpha\beta^2(\alpha+1)$$

$$\Rightarrow \quad Var(X) = E(X^2) - (EX)^2$$

$$= \alpha\beta^2(\alpha+1) - \alpha^2\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$= \boxed{\alpha\beta^2}$$

e $\quad Var(5X+3) = 25\, Var(X) = \boxed{25\,\alpha\beta^2}$

4 $\quad Z$: trt. assignment $\{0 = \text{control}, 1 = \text{trt}\}$

S: infection $\{0 = \text{not}, 1 = \text{infected}\}$

a

$$P(S=1 \mid Z=0) = 0.6$$

$$P(S=1 \mid Z=1) = .5$$

$$P(Z=1) = \tfrac{2}{3}$$

$$P(Z=1 \mid S=1) = \frac{P(S=1 \mid Z=1)\,P(Z=1)}{P(S=1)}$$

$$P(S=1) = P(S=1 \mid Z=1)P(Z=1) + P(S=1 \mid Z=0)P(Z=0)$$

$$= (.5)\left(\tfrac{2}{3}\right) + (.6)\left(\tfrac{1}{3}\right)$$

$$\Rightarrow P(Z=1 \mid S=1) = \boxed{\frac{(.5)\left(\frac{2}{3}\right)}{(.5)\left(\frac{2}{3}\right) + (.6)\left(\frac{1}{3}\right)}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{3}{5}\right)\left(\frac{1}{3}\right)} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{1}{3}}{\frac{5+3}{15}}$$

$$= \frac{15}{8(3)} = \frac{5 \cdot 3}{8 \cdot 3} = \boxed{\frac{5}{8}}$$

$\boxed{b}$

$$P(S_0=1, S_1=1) \le \min(P(S_0=1), P(S_1=1))$$
$$= \min\{0.5, 0.6\}$$
$$= \boxed{0.5}$$

$$P(S_0=1, S_1=1) \ge \max\{P(S_0=1) + P(S_1=1) - 1, 0\}$$
$$= \max\{.5 + .6 - 1, 0\}$$
$$= \boxed{0.1}$$

$\boxed{c}$ $P(S_0=1, S_1=1) = P(S_0=1)P(S_1=1)$
$$= (0.5)(0.6)$$
$$= \boxed{0.3}.$$

No, because if you're infected in one arm you're probably more likely to be infected in other arm.