$\boxed{1}$ $\boxed{a}$ $E(X^{1/2}) \leq (E(X))^{1/2}$ for $x > 0$.

$g(x) = x^{1/2}$

$\Rightarrow g'(x) = \frac{1}{2} x^{-1/2}$

$\Rightarrow g''(x) = -\frac{1}{4} x^{-3/2} < 0$ $\Rightarrow$ $g(x)$ is concave

$\Rightarrow$ $E(g(x)) \leq g(E(x))$ by Jensen's Inequality

$\boxed{b}$ $X \sim Beta(\alpha, \beta) \Rightarrow$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } \alpha, \beta > 0, \; 0 < x < 1$$

$$= \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{c(\theta)} \underbrace{I_{[0,1]}(x)}_{h(x)} \exp\Big[\underbrace{(\alpha-1)}_{w_1(\theta)} \underbrace{\log(x)}_{t_1(x)} + \underbrace{(\beta-1)}_{w_2(\theta)} \underbrace{\log(1-x)}_{t_2(x)}\Big]$$

$\boxed{\text{Yes.}}$

$\boxed{c}$ $X \sim Exp(\theta)$ $\Rightarrow$ $f(x) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$

Location shifted: $f(x - \mu) = \frac{1}{\theta} e^{-\frac{(x-\mu)}{\theta}}$ for $x - \mu > 0$

$$= \boxed{\frac{1}{\theta} e^{-\left(\frac{x-\mu}{\theta}\right)} \quad \text{for } x > \mu}$$

$\boxed{d}$ $E(Y) = E(E(Y|X))$

$$E(E(Y|X)) = \int\left(\int y\, f(y|x)\, dy\right) f(x)\, dx$$

$$= \iint y\, f(x,y)\, dy\, dx$$

$$= \int y \int f(x,y)\, dx\, dy$$

$$= \int y\, f(y)\, dy$$

$$= E(Y) \quad \blacksquare$$

$\boxed{e}$ $P(e^{tX} \geq 1) \quad \leq \quad M_X(t) = E(e^{tX})$

From Chebyshev's inequality:

$$P(g(X) \geq r) \leq \frac{E(g(X))}{r} \qquad \text{if } g(x) > 0,\ r > 0$$

In the above, $g(x) = e^{tX} > 0$

$r = 1.$

$\blacksquare$

$\boxed{2}$

$X \perp Y$, $\quad X, Y \sim \text{Gamma}(\alpha, \beta)$

$$U = \frac{X}{X+Y} \qquad V = X+Y$$

$\boxed{a}$ $f(u,v) = ?$

$$f(x,y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \cdot \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-y/\beta} \qquad \text{for } x, y > 0$$

$$u = \frac{x}{x+y} \Rightarrow u = \frac{v-y}{v-y+y} = \frac{v-y}{v} \Rightarrow u = 1 - \frac{y}{v} \Rightarrow \frac{y}{v} = 1-u \Rightarrow$$

$$y = v(1-u)$$

$$v = x+y \Rightarrow x = v - y \Rightarrow x = v - v(1-u)$$

$$= v - v + uv = uv \Rightarrow x = uv$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$\Rightarrow |J| = |v(1-u) + vu|$$

$$\frac{\partial y}{\partial u} = -v \qquad \frac{\partial y}{\partial v} = 1-u$$

$$= |v - uv + uv| = v$$

$$\Rightarrow f(u,v) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} (uv)^{\alpha-1} e^{-uv/\beta} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} (v(1-u))^{\alpha-1} e^{-v(1-u)/\beta} \, v$$

$$= \frac{1}{\Gamma(\alpha)^2 \beta^{2\alpha}} u^{\alpha-1} (1-u)^{\alpha-1} v^{2(\alpha-1)+1} \exp\left(\frac{-1}{\beta}(uv + v(1-u))\right)$$

$$= \boxed{\frac{1}{\Gamma(\alpha)^2 \beta^{2\alpha}} u^{\alpha-1} (1-u)^{\alpha-1} v^{2\alpha-1} e^{-\frac{v}{\beta}} \qquad \text{for } v > 0,\ 0 < u < 1}$$

$\boxed{2}$

$\boxed{b}$ $f(v) = \int_0^1 f(u,v)\,du$

$$= \frac{1}{\Gamma(\alpha)^2 \beta^{2\alpha}} v^{2\alpha-1} e^{-v/\beta} \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(\alpha+\alpha)} \int_0^1 \underbrace{\frac{\Gamma(\alpha+\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} u^{\alpha-1}(1-u)^{\alpha-1}}_{Beta(\alpha,\alpha)} du$$

$$= \frac{\cancel{\Gamma(\alpha)^2}}{\cancel{\Gamma(\alpha)^2}\,\Gamma(2\alpha)\,\beta^{2\alpha}} v^{2\alpha-1} e^{-v/\beta}$$

$$= \frac{1}{\Gamma(2\alpha)\beta^{2\alpha}} v^{2\alpha-1} e^{-v/\beta} \quad \text{for } v > 0$$

$\Rightarrow \boxed{V \sim Gam(2\alpha, \beta)}$

$\boxed{c}$ $f(u) = \int_0^\infty f(u,v)\,dv$

$$= \frac{1}{\Gamma(\alpha)^2 \cancel{\beta^{2\alpha}}} u^{\alpha-1}(1-u)^{\alpha-1} \frac{\Gamma(2\alpha)\cancel{\beta^{2\alpha}}}{1} \int_0^\infty \frac{1}{\Gamma(2\alpha)\beta^{2\alpha}} v^{2\alpha-1} e^{-v/\beta} dv$$

$$= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} u^{\alpha-1}(1-u)^{\alpha-1} \quad \text{for } 0 < u < 1$$

$\Rightarrow U \sim Beta(\alpha, \alpha)$

$\boxed{d}$ Yes, because $f(u,v) = f(u)f(v)$,

or more simply $f(u,v) = h(u)g(v)$.

- The joint distribution factors.