Bios 312: Modern Regression Analysis
Spring, 2013

Homework #5
March 14, 2013

Written problems due at the beginning of class, Thursday, March 21, 2013.

Regression model theory

Let $Y_i \sim N(\beta_0 + \beta_1 * X_i, \sigma^2)$ as in simple linear regression.

1. Find the maximum likelihood estimates of β_0, β_1, and σ^2. Show your work.

2. Compare the MLEs obtained in (1) to the estimate of obtained for minimizing the residual sums of squares. You do not need to derive the least squares estimates (they are given in the notes).

3. Show that $(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \widehat{\boldsymbol{B}}$ where

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \widehat{\boldsymbol{B}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

and $\hat{\beta}_0$ and $\hat{\beta}_1$ are the MLEs found in (1). (Refer back to matrix algebra notes if needed)

4. Let Y_{avg} be the mean of new observations taken at fixed, known covariate X_{avg} in a simple linear regression model where $Y_{avg} = \widehat{\beta}_0 + \widehat{\beta}_1 X_{avg}$. Find the $Var[Y_{avg}]$. At what value of X_{avg} will $Var[Y_{avg}]$ be smallest?

5. Let Y_{new} be the predicted value of a new Y taken at fixed, known covariate X_{new} in a simple linear regression model where $Y_{new} = \widehat{\beta}_0 + \widehat{\beta}_1 X_{new} + \epsilon$. Find the $Var[Y_{new}]$. At what value of X_{new} will $Var[Y_{new}]$ be smallest?

6. Use the $Var[Y_{avg}]$ to construct a (1-α)% confidence interval for the expected value of Y taken at X_{avg}. Use the $Var[Y_{new}]$ to construct a (1-α)% confidence interval for the predicted value of a new Y at X_{new}. Which of these two confidence intervals will always be wider? Why?