Final Exam, Biostatistics 341

14 December, 2015

Please show all your work and perform all calculations to whatever degree of exactness you are able. This test is closed book and no calculators are allowed.

1. Suppose $X_i \sim^{iid} \mathrm{N}(\mu, \sigma^2)$ for $i = 1, \ldots n$. Let $\overline{X}$ and S^2 denote the sample mean and sample variance, respectively.

(a) What is the distribution of $X_1 + X_2 + X_3$?

(b) What is the distribution of $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \mu)^2$?

(c) What is the distribution of $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \overline{X})^2$?

(d) Let $Y = 2\overline{X} + \frac{n-1}{n}S^2$. What is $\mathrm{E}(Y)$?

(e) What is $\mathrm{Cov}(Y, \overline{X})$?

(f) What is the distribution of $\sqrt{n}(\overline{X} - \mu)/S$?

(g) What is the distribution of $(X_1 + X_2 - 2\mu)/(X_1 - X_2)$?

2. For any two random variables X and Y with finite variances, prove that

(a) $\mathrm{E}(XY)=\mathrm{E}[X\mathrm{E}(Y|X)]$.

(b) $\mathrm{Cov}(X,Y)=\mathrm{Cov}(X,\mathrm{E}(Y|X))$.

(c) X and $Y-$ $\mathrm{E}(Y|X)$ are uncorrelated.

(d) $|\rho| \leq 1$, where ρ is the correlation between X and Y. [Hint: Use the Cauchy-Schwarz Inequality.]

(e) There are more orderings of the letters KENTUCKY than of the letters TENNESSEE. State the number of orderings of each.

3. Let $X_i \sim^{iid} \mathrm{Poisson}(\lambda)$ for $i = 1, \ldots, n$.

(a) Derive the moment generating function for X_1.

(b) Derive the large sample distribution of $\overline{X}$.

(c) Derive the exact distribution of $\overline{X}$.

(d) Derive the large sample distribution of $\log_e(\overline{X} + 1)$.

(e) What is greater, $\mathrm{E}\{\log_e(\overline{X} + 1)\}$ or $\log_e(\mathrm{E}(\overline{X}) + 1)$? Why?

(f) What is the probability mass function of the maximum of X_i? [No need to simplify.]

(g) Let $Z_i = \mathrm{I}(X_i = 0)$, where $\mathrm{I}(\cdot)$ is the indicator function and equals 1 if true and 0 if false. What is the probability mass function of Z_i?

4. Let $X_i \sim \mathrm{Pareto}(\alpha_i, \beta_i)$ for $i = 1$ and 2. Therefore, $f_{X_i}(x) = \beta_i \alpha_i^{\beta_i} x^{-\beta_i - 1}$ for $\alpha_i < x < \infty$, $\alpha_i > 0$, and $\beta_i > 0$. X_1 is independent of X_2.

(a) Are X_1 and X_2 iid? Why or why not?

(b) Is the Pareto distribution an exponential family distribution? Why or why not?

(c) Show whether the Pareto distribution a location family, a scale family, a location-scale family, or none of the them.

(d) What is $\mathrm{E}(X_i)$?

(e) What is $\mathrm{Var}(X_i)$?

(f) Give an algorithm for generating 1000 random variables from a Pareto(α_1, β_1) distribution.

(g) What is the cdf of the minimum of X_1 and X_2?

(h) What is $\mathrm{P}(X_1 < X_2)$?

(i) Suppose $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 1$ and 2. What is the distribution of $U = \log_e(X_1/\alpha) + \log_e(X_2/\alpha)$?

1 a $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\boxed{X_1 + X_2 + X_3 \sim N(3\mu, 3\sigma^2)}$$

because $E(X_1 + X_2 + X_3) = 3\mu$

$Var(X_1 + X_2 + X_3) = 3\sigma^2$

and sum of Normals is normal.

b $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} Z_i^2$ where $Z_i \sim N(0,1)$

$$\Rightarrow Z_i^2 \sim \chi^2_1 \Rightarrow \sum_{i=1}^{n} Z_i \sim \boxed{\chi^2_n}$$

c $\frac{1}{\sigma^2}\sum_{1}^{n}(X_i - \bar{X})^2 = \frac{(n-1)S^2}{\sigma^2} \sim \boxed{\chi^2_{n-1}}$

d $Y = 2\bar{X} + \frac{n-1}{n}S^2$.

$E(Y) = E\left(2\bar{X} + \frac{n-1}{n}S^2\right) = 2E(\bar{X}) + \frac{n-1}{n}E(S^2)$

$$= \boxed{2\mu + \frac{n-1}{n}\sigma^2}$$

e $Cov(Y, \bar{X}) = Cov\left(2\bar{X} + \frac{n-1}{n}S^2, \bar{X}\right)$

$= Cov(2\bar{X}, \bar{X}) + Cov\left(\frac{n-1}{n}S^2, \bar{X}\right)$

$= 2\,Cov(\bar{X}, \bar{X}) + \frac{n-1}{n}Cov(S^2, \bar{X})$

$= 2\,Var(\bar{X}) + \frac{n-1}{n}(0)$

$$= \boxed{\frac{2\sigma^2}{n}}$$

1

f $\boxed{\dfrac{\sqrt{n}\,(\bar{x}-\mu)}{S} \sim t_{n-1}}$

g $\dfrac{2\sqrt{2}\,(X_1+X_2-\mu)}{(X_1-X_2)^2}$; If $n=2$, then $\bar{X} = \dfrac{X_1+X_2}{2}$

$\sqrt{n} = \sqrt{2}$

$$* \; S^2 = \frac{1}{2-1}\left[\left(X_1 - \frac{X_1+X_2}{2}\right)^2 + \left(X_2 - \frac{X_1+X_2}{2}\right)^2\right]$$

$$= \left(\frac{X_1-X_2}{2}\right)^2 + \left(\frac{X_2-X_1}{2}\right)^2$$

$$= 2\left(\frac{X_1-X_2}{2}\right)^2 = \frac{1}{2}(X_1-X_2)^2$$

$$\Rightarrow \frac{\sqrt{n}\,(\bar{X}-\mu)}{S} = \frac{\sqrt{2}\left(\frac{X_1+X_2}{2}-\mu\right)}{\sqrt{\frac{1}{2}(X_1-X_2)^2}} \sim t_{2-1}$$

$$= \frac{2\left(\frac{X_1+X_2}{2}-\mu\right)}{X_1-X_2}$$

$$= \frac{2\left(\frac{X_1+X_2-2\mu}{2}\right)}{X_1-X_2}$$

$$= \frac{X_1+X_2-2\mu}{X_1-X_2}$$

$t_1 = \boxed{\text{Cauchy}(0)}$

$\boxed{2}$ $\boxed{a}$ $E(XY) = E\left[X\,E(Y|X)\right]$

$$= \iint x\left(\int y f(y|x)\,dy\right) f(x,y)\,dx\,dy$$

$$= \int\left[x\left(\int y f(y|x)\,dy\right)\underbrace{\int f(y|x)\,dy}_{=1}\right] f(x)\,dx$$

$$= \int x\left(\int y f(y|x)\,dy\right) f(x)\,dx$$

$$= \iint xy\, f(x,y)\,dx\,dy$$

$$= E(XY)$$

$\boxed{b}$ $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, E(Y|X))$

$$= E(X\,E(Y|X)) - E(X)E(E(Y|X))$$

$$= E(XY) \text{ ← from (a)} - E(X)E(Y)$$

$$= \mathrm{Cov}(X,Y)$$

$\boxed{c}$ $\mathrm{Cov}(X, Y - E(Y|X)) = \mathrm{Cov}(X,Y) - \mathrm{Cov}(X, E(Y|X))$

$$= \mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Y) \text{ ← from (b)}$$

$$= 0$$

$\boxed{2}$

$\boxed{d}$ $|\rho| \leq 1$

Cauchy-Schwarz Inequality: $|E(XY)| \leq E|XY| \leq \left(E|X|^2\right)^{1/2}\left(E|Y|^2\right)^{1/2}$

$$|\mathrm{cov}(X,Y)| = \left|E\left[(X-\mu_x)(Y-\mu_y)\right]\right| \leq \left(E\left[|X-\mu_x|^2\right]\right)^{1/2}\left(E\left[|Y-\mu_y|^2\right]\right)^{1/2}$$

$$= \mathrm{Var}(X)^{1/2}\ \mathrm{Var}(Y)^{1/2}$$

$$\Rightarrow \quad \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \leq 1$$

$$\Rightarrow \quad |\rho| \leq 1$$

■

$\boxed{e}$ TENNESSEE

$$\frac{9!}{4!\,2!\,2!}$$

9! ways to order 9 letters
4! ways to order 4 E's
2! ways to order 2 S's &
2! ways to order 2 N's

} divide by this so not over-counting.

KENTUCKY

$$\frac{8!}{2!}$$

Comparing: $\dfrac{8!}{\cancel{2!}} \geq \dfrac{9!}{4!\,2!\,\cancel{2!}} = \dfrac{9 \cdot 8!}{4!\,2!}$

because $1 \geq \dfrac{9}{48}$

3 $X_i \sim \text{Poisson}(\lambda)$

a $$E(e^{tx}) = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda} \lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} \sum_{x=0}^{\infty} \frac{e^{-(\lambda e^t)} (\lambda e^t)^x}{x!}$$

$$= \boxed{e^{\lambda(e^t - 1)}}$$

b $E(X_1) = \lambda = \mu$

$Var(X_1) = \lambda = \sigma^2$

$\Rightarrow \sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$ by CLT

$\Rightarrow \boxed{\sqrt{n}(\bar{X} - \lambda) \xrightarrow{d} N(0, \lambda)}$

c $\bar{X} = \frac{1}{n} \sum X_i = \frac{1}{n} Y$

$$M_Y(t) = E[e^{tY}] = E[e^{\sum X_i t}] = E[\prod e^{X_i t}] = \prod E(e^{X_i t})$$

$$= \prod M_{X_i}(t) = (M_{X_1}(t))^n = (e^{\lambda(e^t - 1)})^n$$

$$= e^{n\lambda(e^t - 1)} = \text{mgf for Poisson}(n\lambda)$$

$\Rightarrow \sum X_i \sim \text{Poisson}(n\lambda)$

$\Rightarrow \boxed{n\bar{X} \sim \text{Poisson}(n\lambda)}$

3

d $\sqrt{n}(\bar{X}-\lambda) \xrightarrow{d} N(0,\lambda)$ by (b)

$g(\mu) = \log(\mu+1)$

$\Rightarrow g'(\mu) = \frac{1}{\mu+1}$

$\Rightarrow$ by Δ-method that

$$\boxed{\sqrt{n}\left(\log(\bar{X}+1) - \log(\lambda+1)\right) \xrightarrow{d} N\left(0, \frac{\lambda}{(\lambda+1)^2}\right)}$$

e $g(\mu) = \log(\mu+1)$

$g'(\mu) = (\mu+1)^{-1}$

$g''(\mu) = -1(\mu+1)^{-2} < 0$

$\Rightarrow \log(\mu+1)$ is a concave function

$\Rightarrow \boxed{E\left[\log(\bar{X}+1)\right] \leq \log\left[E(\bar{X})+1\right]}$ by Jensen's Inequality

f $P(X_{(n)} \leq y) = P(\text{All } X \leq y)$

$= \left[P(X_1 \leq y)\right]^n = \left[\sum_{t=0}^{y} \frac{e^{-\lambda}\lambda^t}{t!}\right]^n$

$P(X_{(n)} = y) = P(X_{(n)} \leq y) - P(X_{(n)} \leq y-1)$

$$= \boxed{\left[\sum_{t=0}^{y} \frac{e^{-\lambda}\lambda^t}{t!}\right]^n - \left[\sum_{t=0}^{y-1} \frac{e^{-\lambda}\lambda^t}{t!}\right]^n}$$

$\boxed{3}$ $\boxed{9}$ $Z_i = I(X_i = 0)$

$Z_i = 1$ with probability $p = P(X_i = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$

$Z_i = 0$ w/ probability $1-p = 1-e^{-\lambda}$

$$\Rightarrow \boxed{Z_i \overset{iid}{\sim} \text{Bernoulli}(e^{-\lambda})}$$

$\boxed{4}$ $X_i \sim \text{Pareto}(\alpha_i, \beta_i)$ for $i = 1, 2$

$f_{X_1}(x) = \beta_1 \alpha_1^{\beta_1} x^{-\beta_1 - 1}$ $\quad \alpha_1 < x < \infty$

$\alpha_1, \beta_1 > 0$

(a) X_1 and X_2 are not iid because they are not identically distributed. They have different parameters (presumably).

(b) No, because the support is a function of the parameter α_1.

$\boxed{c}$ Write $f(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}} I_{(\alpha,\infty)}(x)$

$$\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right) = \frac{1}{\sigma} \frac{\beta \alpha^\beta}{\left(\frac{x}{\sigma}\right)^{\beta+1}} I_{(\alpha,\infty)}(x/\sigma)$$

$$= \frac{\sigma^{\beta+1}}{\sigma} \frac{\beta \alpha^\beta}{x^{\beta+1}} I_{(\sigma\alpha,\infty)}(x)$$

$$= \frac{\beta (\alpha\sigma)^\beta}{x^{\beta+1}} I_{(\sigma\alpha,\infty)}(x) = \text{pdf for Pareto}(\sigma\alpha, \beta),$$

so it is a scale family.

$$f(x-\mu) = \frac{\beta \alpha^\beta}{(x-\mu)^{\beta+1}} I_{(\alpha,\infty)}(x-\mu) = \frac{\beta \alpha^\beta}{(x-\mu)^{\beta+1}} I_{(\alpha+\mu,\infty)}(x)$$

extra μ, so this is not the pdf for a Pareto distribution.

$\Rightarrow$ Not location family.

Since not a location family, it is also not a location-scale family.

$\Rightarrow$ $\boxed{\text{Pareto is a scale family}}$

$\boxed{4}$

$\boxed{d}$ $$E(X) = \int_{\alpha}^{\infty} x\,\beta\alpha^{\beta}x^{-\beta-1}\,dx = \int_{\alpha}^{\infty}\beta\alpha^{\beta}x^{-\beta}\,dx$$

$$= \left.\frac{\beta\alpha^{\beta}x^{-\beta+1}}{-\beta+1}\right]_{\alpha}^{\infty} = 0 - \frac{\beta\alpha^{\beta}\alpha^{1-\beta}}{1-\beta}$$

$$= \boxed{\frac{\beta}{\beta-1}\alpha}$$

$\boxed{e}$ $$E(X^2) = \int_{\alpha}^{\infty}\beta\alpha^{\beta}x^{-\beta+1}\,dx = \left.\frac{\beta\alpha^{\beta}x^{-\beta+2}}{-\beta+2}\right]_{\alpha}^{\infty}$$

$$= 0 - \frac{\beta\alpha^{\beta}\alpha^{-\beta+2}}{2-\beta} = \frac{\beta}{\beta-2}\alpha^2$$

$$\Rightarrow \mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$= \frac{\beta}{\beta-2}\alpha^2 - \left(\frac{\beta}{\beta-1}\right)^2\alpha^2$$

$$= \alpha^2\left[\frac{\beta(\beta-1)^2 - \beta^2(\beta-2)}{(\beta-1)^2(\beta-2)}\right]$$

$$= \beta(\beta^2-2\beta+1) - \beta^3 + 2\beta^2$$
$$= \beta^3 - 2\beta^2 + \beta - \beta^3 + 2\beta^2$$
$$= \beta$$

$$= \boxed{\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}}$$

$\boxed{4}$

$\boxed{f}$ $$F(x) = \int_{\alpha_1}^{x} \beta_1 \alpha_1^{\beta_1} x_1^{-\beta_1 - 1} dx_1$$

$$= \frac{\beta_1 \alpha_1^{\beta_1}}{-\beta_1} x_1^{-\beta_1}\Big]_{\alpha_1}^{x} = -\alpha_1^{\beta_1} x^{-\beta_1} + \alpha_1^{\beta_1 - \beta_1} = 1 - \left(\frac{\alpha_1}{x}\right)^{\beta_1}$$

$F_x(X) \sim \text{Unif}(0,1)$

$$\Rightarrow u = 1 - \left(\frac{\alpha_1}{x}\right)^{\beta_1} \Rightarrow 1 - u = \left(\frac{\alpha_1}{x}\right)^{\beta_1}$$

$$\Rightarrow (1-u)^{1/\beta_1} = \alpha_1 / x \Rightarrow x = \alpha_1 / (1-u)^{1/\beta_1} = \alpha_1 (1-u)^{-1/\beta_1}$$

Generate 1000 $U \sim \text{Unif}(0,1)$.

Let $X = \alpha_1 (1-U)^{-1/\beta_1}$.

$\boxed{g}$ $$P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$$

$$= 1 - P(X_1 > x, X_2 > x)$$

$$= 1 - P(X_1 > x) P(X_2 > x)$$

$$= 1 - (1 - F_{X_1}(x))(1 - F_{X_2}(x))$$

$$= 1 - \left[1 - \left(1 - \left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)\right]\left[1 - \left(1 - \left(\frac{\alpha_2}{x}\right)^{\beta_2}\right)\right]$$

$$= 1 - \left(+\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right)\left(+\left(\frac{\alpha_2}{x}\right)^{\beta_2}\right)$$

$\boxed{4}$

$\boxed{h}$ (continued):

If $\alpha_1 < \alpha_2$:

$$P(X_1 < X_2) = \int_{\alpha_2}^{\infty}\int_{\alpha_1}^{\alpha_2} \beta_1 \alpha_1^{\beta_1} x_1^{-\beta_1-1} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_1\, dx_2$$

$$+ \int_{\alpha_2}^{\infty}\int_{\alpha_2}^{x_2} \beta_1 \alpha_1^{\beta_1} x^{-\beta_1-1} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_1\, dx_2$$

$$= \int_{\alpha_2}^{\infty} \frac{\beta_1 \alpha_1^{\beta_1} x^{-\beta_1}}{-\beta_1}\Big|_{\alpha_1}^{\alpha_2} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_2 + \int_{\alpha_2}^{\infty} -\alpha_1^{\beta_1} x^{-\beta_1}\Big|_{\alpha_2}^{x_2} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_2$$

$$= \int_{\alpha_2}^{\infty} \left(1 - \alpha_1^{\beta_1}\alpha_2^{-\beta_1}\right) \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_2 + \int_{\alpha_2}^{\infty} \left(\alpha_1^{\beta_1}\alpha_2^{-\beta_1} - \alpha_1^{\beta_1} x_2^{-\beta_1}\right) \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_2$$

$$= -x_2^{-\beta_2}\alpha_2^{\beta_2}\Big|_{\alpha_2}^{\infty}\left(1 - \alpha_1^{\beta_1}\alpha_2^{-\beta_1}\right) - \alpha_1^{\beta_1}\alpha_2^{-\beta_1}\alpha_2^{\beta_2} x_2^{-\beta_2}\Big|_{\alpha_2}^{\infty} - \frac{\alpha_1^{\beta_1}\beta_2\alpha_2^{\beta_2} x_2^{-\beta_2-\beta_1}}{-\beta_2-\beta_1}\Big|_{\alpha_2}^{\infty}$$

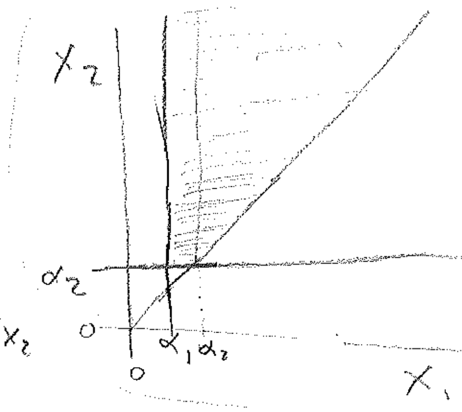
$$= \alpha_2^{-\beta_2}\alpha_2^{\beta_2}\left(1 - \alpha_1^{\beta_1}\alpha_2^{-\beta_1}\right) + \alpha_1^{\beta_1}\alpha_2^{-\beta_1} - \frac{\beta_2}{\beta_2+\beta_1}\alpha_1^{\beta_1}\alpha_2^{\beta_2}\alpha_2^{-\beta_2-\beta_1}$$

$$= 1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{\beta_1} + \left(\frac{\alpha_1}{\alpha_2}\right)^{\beta_1} - \frac{\beta_2}{\beta_1+\beta_2}\left(\frac{\alpha_1}{\alpha_2}\right)^{\beta_1}$$

$$= 1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{\beta_1}\frac{\beta_2}{\beta_1+\beta_2} \quad \text{if } \alpha_2 > \alpha_1.$$

$\boxed{4}$

$\boxed{h}$ $P(X_1 < X_2)$



Assuming $\alpha_1 < \alpha_2$

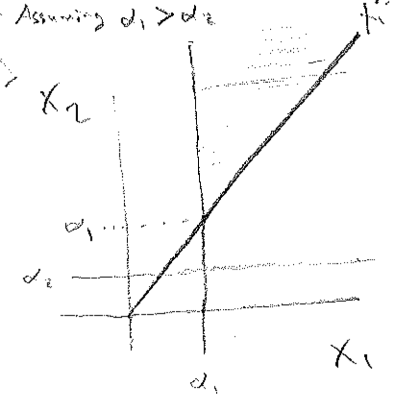
$f(x_1,x_2)$: $\beta_1 \alpha_1^{\beta_1} x_1^{-\beta_1-1} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}$

Assuming $\alpha_1 > \alpha_2$:

$$P(X_1 < X_2) = \int_{\alpha_1}^{\infty}\int_{\alpha_1}^{x_2} \beta_1 \alpha_1^{\beta_1} x_1^{-\beta_1-1} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_1\, dx_2$$

$$= \int_{\alpha_1}^{\infty} \frac{\beta_1 \alpha_1^{\beta_1} x_1^{-\beta_1}}{-\beta_1}\Bigg]_{\alpha_1}^{x_2} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_2$$

Assuming $\alpha_1 > \alpha_2$



$$= \int_{\alpha_1}^{\infty} \alpha_1^{\beta_1}\left[\alpha_1^{-\beta_1} - x_2^{-\beta_1}\right] \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_1-1}\, dx_2$$

$$= \int_{\alpha_1}^{\infty} \beta_2 \alpha_2^{\beta_2} x_2^{-\beta_2-1}\, dx_2 - \int_{\alpha_1}^{\infty} \beta_2 \alpha_1^{\beta_1} x_2^{-\beta_1-\beta_2-1} \alpha_2^{\beta_2}\, dx_2$$

$$= -\alpha_2^{\beta_2} x_2^{-\beta_2}\Bigg]_{\alpha_1}^{\infty} - \frac{\alpha_2^{\beta_2}\alpha_1^{\beta_1} \beta_2 x_2^{-\beta_1-\beta_2}}{-\beta_1-\beta_2}\Bigg]_{\alpha_1}^{\infty}$$

$$= \alpha_2^{\beta_2} \alpha_1^{-\beta_2} - \frac{\alpha_1^{\beta_1}\beta_2 \alpha_1^{-\beta_1-\beta_2} \alpha_2^{\beta_2}}{\beta_1+\beta_2}$$

$$= \left(\frac{\alpha_2}{\alpha_1}\right)^{\beta_2} - \frac{\beta_2}{\beta_1+\beta_2}\left(\frac{\alpha_2}{\alpha_1}\right)^{\beta_2} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\beta_2}\left(1 - \frac{\beta_2}{\beta_1+\beta_2}\right) \quad \text{if } \alpha_1 > \alpha_2.$$

$$= \left(\frac{\alpha_2}{\alpha_1}\right)^{\beta_2}\left(\frac{\beta_1}{\beta_1+\beta_2}\right)$$

$\boxed{4}$ $\boxed{i}$

$$U = \log\left(\frac{X_1}{\alpha}\right) + \log\left(\frac{X_2}{\alpha}\right) \Rightarrow u = v + \log\left(\frac{x_2}{\alpha}\right) \Rightarrow x_2 = \alpha e^{u-v}$$

$$V = \log\left(\frac{X_1}{\alpha}\right) \Rightarrow X_1 = \alpha e^{v}$$

$$\Rightarrow \frac{\partial x_1}{\partial u} = 0 \qquad \frac{\partial x_1}{\partial v} = \alpha e^{v}$$

$$\frac{\partial x_2}{\partial u} = \alpha e^{u-v} \qquad \frac{\partial x_2}{\partial v} = -\alpha e^{u-v}$$

$$\Rightarrow J = \alpha^2 e^{u}$$

$$\Rightarrow f(u,v) = \beta\alpha^{\beta}(\alpha e^{v})^{-\beta-1}\,\beta\alpha^{\beta}(\alpha e^{u-v})^{-\beta-1}\,\alpha^2 e^{u}$$

$$= \beta^2 \alpha^{\beta-\beta-1+\beta-\beta-1+2}\, e^{-v\beta-v+v\beta+v}\, e^{u-u\beta-u}$$

$$= \beta^2 e^{-u\beta} \quad \text{for} \quad 0 < v < u < \infty$$

$$\Rightarrow f(u) = \int_0^u \beta^2 e^{-u\beta}\,dv = \beta^2 u e^{-u\beta} \quad \text{for} \quad 0 < u < \infty$$

$$\Rightarrow \boxed{U \sim \text{Gamma}(2, 1/\beta)}$$